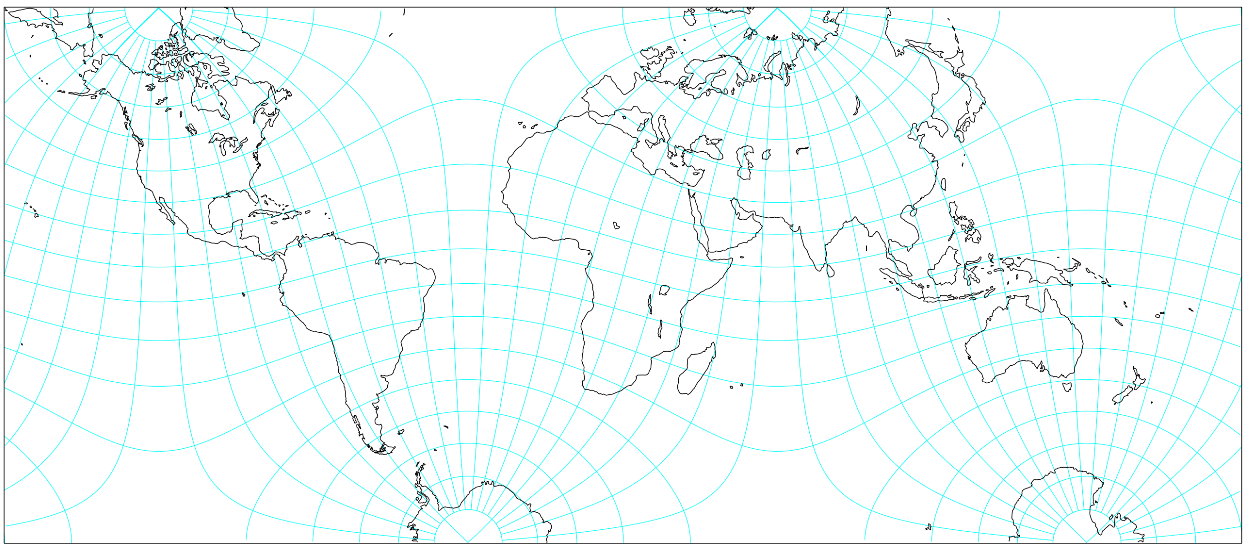
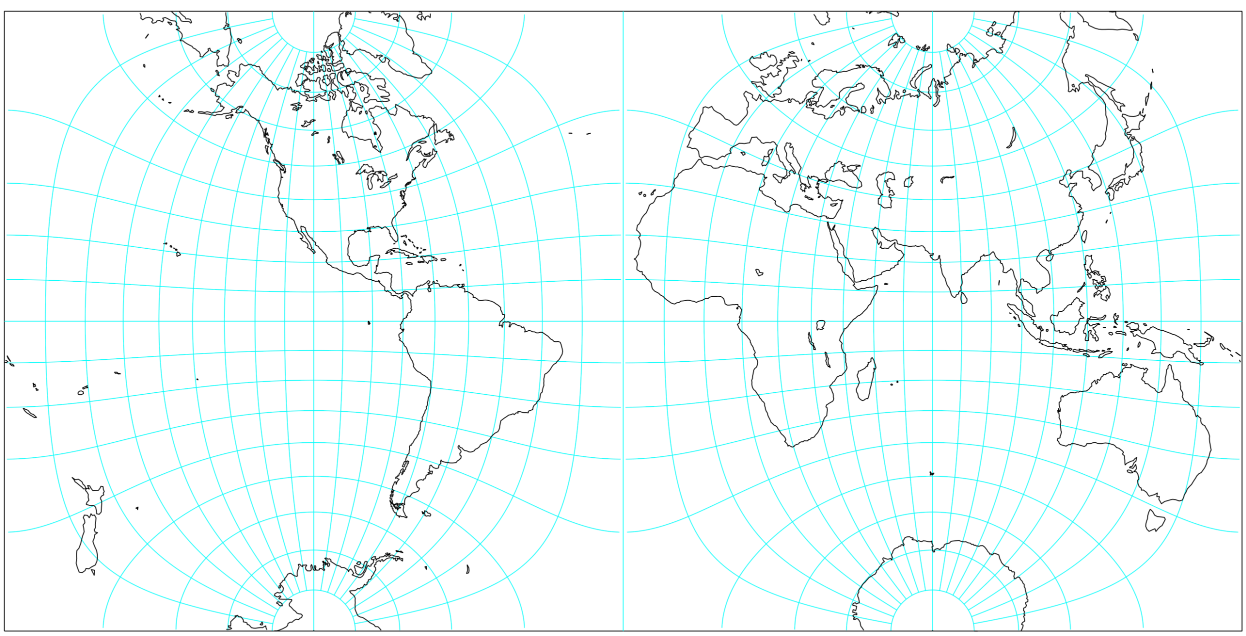
In 1982, I invented a conformal projection of the full earth onto a tetrahedron as an improvement on a projection invented by Emile Guyou in 1886. My motivation was the observation that his projection has four singularities spaced 90° apart on a great circle through the poles (at the corners of the hemispheres on the Guyou map below), so any singularity is 90° away from two of the other singularities and 180° away from the fourth one. I thought that a better projection might result from locating the four singularities at the vertices of a regular tetrahedron lying inside the globe so that each one is an equal distance $109.47^{\circ} = \cos^{-1}(-1/3)$ from the other three. Using this projection to construct a tetrahedral "globe" is entertaining, but the goal of a full-earth projection is a flat two-dimensional world map (somewhat quaint in this digital era), not a mapping onto a three-dimensional figure. My map and Guyou's map are displayed for comparison below:



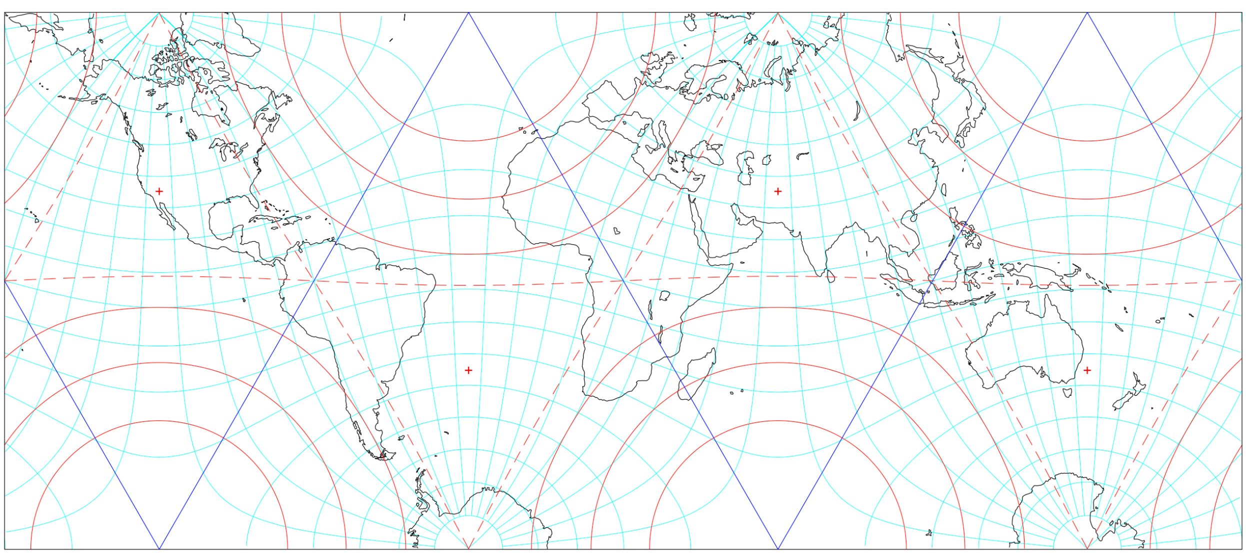
Tetrahedral Projection



Guyou Projection

The four singularities of my map are at latitudes ±35.26439° = ± (1/2) $\cos^{-1}(1/3)$, with the northern ones at longitudes 25° W and 155° E and the southern ones at longitudes 115° W and 65° E. The northern hemisphere has a similar appearance on the tetrahedral and Guyou maps, but the southern hemisphere looks quite different. Australia and New Zealand, in particular, have much less distortion on the tetrahedral map. The shortest distance between the north and south poles on both maps is equal to half the east-to-west extension of the map, as it is on the globe. It is interesting, but probably not significant, that the sum of the cosines of the distances between any singularity and the other three is –1 for both maps. Some people will find it disturbing that no parallel or meridian on the tetrahedral map is a straight line.

I worked out the math, programmed it in Basic, and plotted a few maps in 1982. I learned during this time that H. A. Schwarz (1869) and L. P. Lee (1965) had invented the tetrahedral projection previously, but Schwarz didn't apply it to terrestrial cartography, and Lee put one singularity at the north pole and three evenly spaced on the 19.47° S parallel. I set this aside until 2019 when I found a file of coastline data points and a paper by M. Douglas McIlroy, who had also invented the tetrahedral projection circa 1976 (https://www.cs.dartmouth.edu/~doug/wallpaper.pdf). My original code was lost and my old notes weren't easy to decipher, so I re-programmed everything in Matlab on a MacBook Pro using used McIlroy's equations. The new code and faster computer create a world map much more rapidly than the software and hardware I used in 1962, making it easy to investigate variations of the parameters of the map, including exactly where to locate its singularities and where to set its boundaries. I settled on the final singularity locations after vacillating between placing them five degrees to the east or west. The resulting tetrahedral map and Guyou's map are shown with added notations here and on the next page.



The faces of the tetrahedron are the four equilateral triangles bounded by the blue lines and the upper and lower boundaries of the map. The points with minimum magnification, marked by the red plus signs, are at the centers of the faces of the tetrahedron. The magnification is infinite at the singularities, where conformality fails. Let s denote the ratio of the area magnification (the square of the linear magnification) at any point to the minimum area magnification on the map. The area magnification ratio along the dashed red lines is $s = 2/\sqrt{3}$. The area magnification ratios on the solid red lines are $s = 4/3$, 2, and 4, the Mercator projection's ratios at latitudes ±30°, ±45°, and ±60°, respectively. The map's center is on the equator at longitude 20° E.

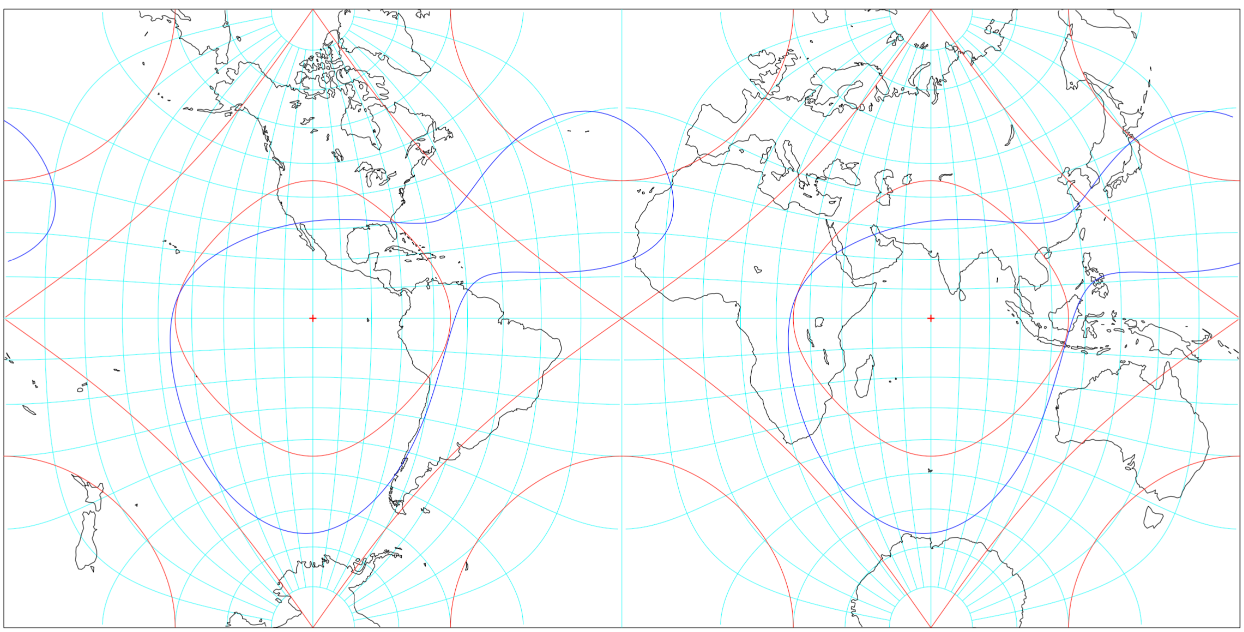
The Mercator projection has $1 \le s \le 4/3$ for 0.500 of the globe, $4/3 \le s \le 2$ for 0.207 of the globe, $2 \le s \le 4$ for 0.159 of the globe, and $4 \le s$ for 0.134 of the globe. For the tetrahedral projection, the globe fractions and areas on the map with these magnification ranges are:

magnification range	fraction of globe	area on map	average s	fraction of map
$1 \le s \le 4/3$	0.620	$0.710\,\mu$	1.144	0.437
$4/3 \le s \le 2$	0.241	$0.379\,\mu$	1.576	0.233
$2 \le s \le 4$	0.107	$0.283\,\mu$	2.642	0.174
$4 \le s$	0.032	$0.254\,\mu$	7.936	0.156
total	1.000	$1.62596\,\mu$	1.62596	1.000

where μ denotes the minimum area magnification. The statistics for Guyou's projection are:

magnification range	fraction of globe	area on map	average s	fraction of map
$1 \le s \le 4/3$	0.270	$0.312\,\mu$	1.155	0.142
$4/3 \le s \le 2$	0.379	$0.631\,\mu$	1.663	0.288
$2 \le s \le 4$	0.285	$0.729\,\mu$	2.556	0.333
$4 \le s$	0.066	$0.517\,\mu$	7.866	0.236
total	1.000	$2.18844\,\mu$	2.18844	1.000

Accounting for the map areas, this gives $\mu_{\text{tetra}}/\mu_{\text{Guyou}} = (\sqrt{3}/2)(2.18844/1.62596) = 1.1656$ for tetrahedral and Guyou maps having the same width.



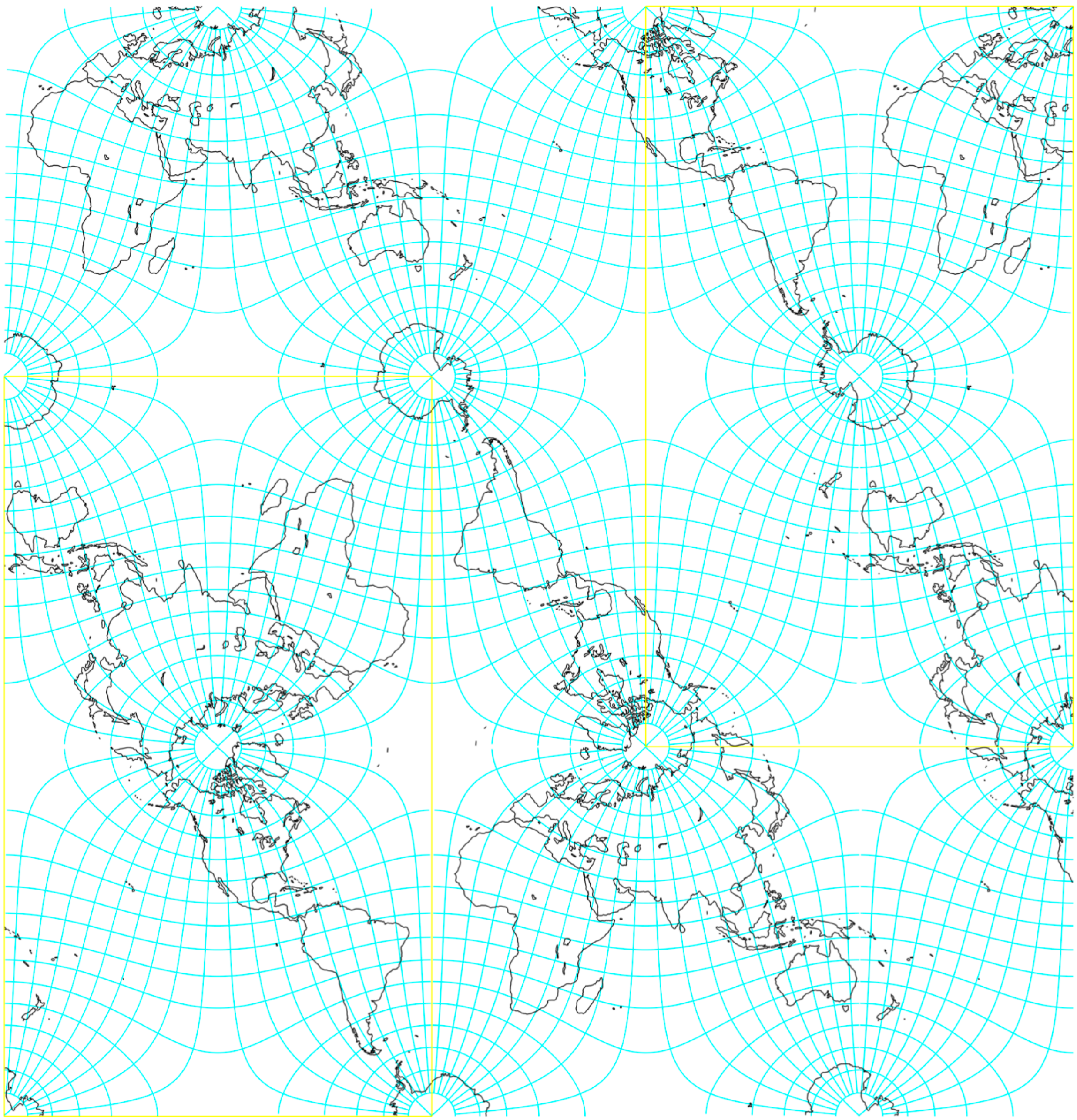
The red plus signs and solid red lines on the Guyou map above have the same meaning as on the tetrahedral map. The blue lines identify the points where a Guyou map and a tetrahedral map of the same width have equal area magnifications (the products of μ and s).

The equator and the two meridians midway between the Guyou map's singularities are straight lines. The equator and the four meridians midway between the tetrahedral map's singularities are the great circle segments on that map that stay the closest (within ±15° in direction) to straight lines connecting their endpoints. The variation of the magnification along these great circles is much less on the tetrahedral map ($2/\sqrt{3} \le s \le 4/3$) than on Guyou's map ($1 \le s \le 2$).

The only significant land areas with area magnification ratio $s > 4$ on the tetrahedral map are:

in the Azores	$s = 24.0$ at Vila do Porto, 36.946° N, 25.148° W = Mercator s at latitude ±78.22°
in Madeira	$s = 5.68$ at Funchal, 37.667° N, 16.924° W
in the Canaries	$s = 4.55$ at Santa Cruz de la Palma, 28.684° N, 17.765° W
Easter Island	$s = 4.33$ at 27.113° S, 109.350° W

The tetrahedral and Guyou maps can be infinitely repeated to tile the plane, as the plot below illustrates for the tetrahedral projection. The tetrahedral map shown elsewhere in this note is the central 4/5 of the bottom 1/3 of this plot. This plot also contains rectangular world maps with the north or south pole at the center, the other pole at the corners, and a singularity at the midpoint of each side; two of these polar maps are outlined in yellow.



Appendix: Equations for computing map areas and fractions

The ratio of the area magnification at any point to the minimum area magnification on the tetrahedral map is given by the equation

$$s_{\text{tetra}} = 4/\sqrt{27(1-\cos\alpha)(1-\cos\beta)(1-\cos\gamma)(1-\cos\delta)}$$

where α, β, γ, and δ are the arc length distances from the point to the four singularities.

The ratio for the Guyou projection is

$$s_{\text{Guyou}} = 1/\sqrt{(1-\cos\alpha)(1-\cos\beta)(1-\cos\gamma)(1-\cos\delta)}$$

The ratio for the Mercator projection, which has singularities only at the north and south poles, is

$$s_{\text{Mercator}} = 1/[(1-\cos\alpha)(1-\cos\beta)] = 1/\cos^2\theta,$$

where θ is the latitude at the point.

The total area of the tetrahedral map (more precisely, the ratio of this area to the total surface area of the sphere being mapped) is

$$(2/\pi)[K(m_{\text{tetra}})]^2\mu_{\text{tetra}} = 1.62596\,\mu_{\text{tetra}}$$

where $K(m)$ is the complete elliptic integral of the first kind with parameter m,

$$m_{\text{tetra}} = \sin^2(\pi/12) = (2-\sqrt{3})/4$$

and μ_{tetra} is the minimum area magnification on the map. The argument of the sine function appearing in m is known as the modular angle.

The total area of the Guyou map is

$$(2/\pi)[K(m_{\text{Guyou}})]^2\mu_{\text{Guyou}} = 2.18844\,\mu_{\text{Guyou}}$$

where

$$m_{\text{Guyou}} = \sin^2(\pi/4) = 1/2$$

It is well known that the total area of the Mercator map is infinite.